ECE 604, Lecture 6

Jan 18, 2019

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Printed on March 24, 2019 at 16 : 02: W.C. Chew and D. Jiao.

1 Time-Harmonic Fields—Linear Systems

The analysis of Maxwell's equations can be greatly simplified by assuming the fields to be time harmonic, or sinusoidal (cosinusoidal). Electrical engineers use a method called phasor technique to simplify equations involving time-harmonic signals. This is also a poor-man's Fourier transform. That is one begets the benefits of Fourier transform technique without knowledge of Fourier transform. Since only a time-harmonic frequency is involved, this is also called frequency domain analysis.[1]

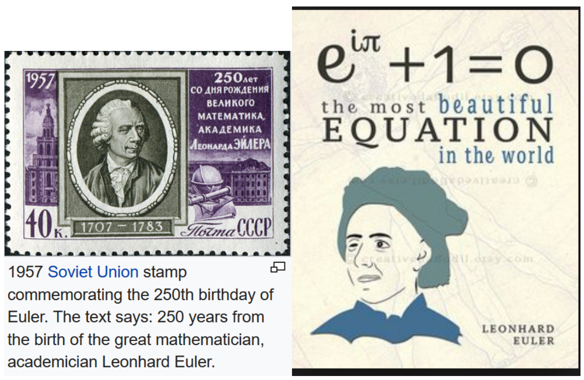

Figure 1: Courtesy of Wikipedia and Pinterest.

To learn phasor techniques, one makes use the formula due to Euler (1707–1783)

$$e^{j\alpha} = \cos\alpha + j\sin\alpha \tag{1.1}$$

where $j = \sqrt{-1}$ is an imaginary number. But lo and behold, in other disciplines, $\sqrt{-1}$ is denoted by "i", but "i" is too close to the symbol for current. So the preferred symbol for electrical engineering for an imaginary number is j: a quirkness of convention, just as positive charges do not carry current in a wire.

From Euler's formula one gets

$$\cos\alpha = \Re e(e^{j\alpha}) \tag{1.2}$$

Hence, all time harmonic quantity can be written as

$$\begin{aligned} V(x,y,z,t) &= V'(x,y,z)\cos(\omega t+\alpha) && (1.3)\\ &= V'(\mathbf{r})\Re e(e^{j(\omega t+\alpha)}) && (1.4)\\ &= \Re e\left(V'(\mathbf{r})e^{j\alpha}e^{j\omega t}\right) && (1.5)\\ &= \Re e\left(\underset{\sim}{V}(\mathbf{r})e^{j\omega t}\right) && (1.6)\end{aligned}$$

[1]It is simple only for linear systems: for nonlinear systems, such analysis can be quite unwieldy. But rest assured, as we will not discuss nonlinear systems in this course.

Now $\underset{\sim}{V}(\mathbf{r}) = V'(\mathbf{r})e^{j\alpha}$ is a complex number called the phasor representation of $V(\mathbf{r}, t)$ a time-harmonic quantity.[2] Consequently, any component of a field can be expressed as

$$E_x(x, y, z, t) = E_x(\mathbf{r}, t) = \Re e[\underset{\sim}{E}_x(\mathbf{r})e^{j\omega t}] \tag{1.7}$$

The above can be repeated for y and z components. Compactly, one can write

$$\mathbf{E}(\mathbf{r}, t) = \Re e[\underset{\sim}{\mathbf{E}}(\mathbf{r})e^{j\omega t}] \tag{1.8}$$

$$\mathbf{H}(\mathbf{r}, t) = \Re e[\underset{\sim}{\mathbf{H}}(\mathbf{r})e^{j\omega t}] \tag{1.9}$$

where $\underset{\sim}{\mathbf{E}}$ and $\underset{\sim}{\mathbf{H}}$ are complex vector fields. Such phasor representations of time-harmonic fields simplify Maxwell's equations. For instance, if one writes

$$\mathbf{B}(\mathbf{r}, t) = \Re e\left(\underset{\sim}{\mathbf{B}}(\mathbf{r})e^{j\omega t}\right) \tag{1.10}$$

then

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}, t) &= \frac{\partial}{\partial t}\Re e[\underset{\sim}{\mathbf{B}}(\mathbf{r})e^{j\omega t}] \\
&= \Re e\left(\frac{\partial}{\partial t}\underset{\sim}{\mathbf{B}}(\mathbf{r})j\omega e^{j\omega t}\right) \\
&= \Re e\left(\underset{\sim}{\mathbf{B}}(\mathbf{r})j\omega e^{j\omega t}\right)
\end{aligned} \tag{1.11}$$

Therefore, given Faraday's law that

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{M} \tag{1.12}$$

assuming that all quantities are time harmonic, then

$$\mathbf{E}(\mathbf{r}, t) = \Re e[\underset{\sim}{\mathbf{E}}(\mathbf{r})e^{j\omega t}] \tag{1.13}$$

$$\mathbf{M}(\mathbf{r}, t) = \Re e[\underset{\sim}{\mathbf{M}}(\mathbf{r})e^{j\omega t}] \tag{1.14}$$

using (1.11), and (1.14), into (1.12), one gets

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = \Re e[\nabla \times \underset{\sim}{\mathbf{E}}(\mathbf{r})e^{j\omega t}] \tag{1.15}$$

and that

$$\Re e[\nabla \times \underset{\sim}{\mathbf{E}}(\mathbf{r})e^{j\omega t}] = -\Re e[\underset{\sim}{\mathbf{B}}(\mathbf{r})j\omega e^{j\omega t}] - \Re e[\underset{\sim}{\mathbf{M}}(\mathbf{r})e^{j\omega t}] \tag{1.16}$$

[2]We will use under tilde to denote a complex number or a phasor here, but this notation will be dropped later. Whether a variable is complex or real is clear from the context.

Since if

$$\Re e[Ae^{j\omega t}] = \Re e[B(\mathbf{r})e^{j\omega t}], \quad \forall t \tag{1.17}$$

then $A = B$, it must be true from (1.16) that

$$\nabla \times \underset{\sim}{\mathbf{E}}(\mathbf{r}) = -j\omega \underset{\sim}{\mathbf{B}}(\mathbf{r}) - \underset{\sim}{\mathbf{M}}(\mathbf{r}) \tag{1.18}$$

Hence, finding the phasor representation of an equation is clear: whenever we have $\frac{\partial}{\partial t}$, we replace it by $j\omega$. Applying this methodically to the other Maxwell's equations, we have

$$\nabla \times \underset{\sim}{\mathbf{H}}(\mathbf{r}) = j\omega \underset{\sim}{\mathbf{D}}(\mathbf{r}) + \underset{\sim}{\mathbf{J}}(\mathbf{r}) \tag{1.19}$$

$$\nabla \cdot \underset{\sim}{\mathbf{D}}(\mathbf{r}) = \varrho_e(\mathbf{r}) \tag{1.20}$$

$$\nabla \cdot \underset{\sim}{\mathbf{B}}(\mathbf{r}) = \varrho_m(\mathbf{r}) \tag{1.21}$$

2 Fourier Transform Technique

In the phasor representation, Maxwell's equations has no time derivatives; hence the equations are simplified. We can also arrive at the above simplified equations using Fourier transform technique. To this end, we use Faraday's law as an example. By letting

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, \omega) e^{j\omega t} d\omega \tag{2.1}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{B}(\mathbf{r}, \omega) e^{j\omega t} d\omega \tag{2.2}$$

$$\mathbf{M}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{M}(\mathbf{r}, \omega) e^{j\omega t} d\omega \tag{2.3}$$

Substituting the above into Faraday's law given by (1.12), we get

$$\nabla \times \int_{-\infty}^{\infty} d\omega e^{j\omega t} \mathbf{E}(\mathbf{r}, \omega) = -\frac{\partial}{\partial t} \int_{-\infty}^{\infty} d\omega e^{j\omega t} \mathbf{B}(\mathbf{r}, \omega) - \int_{-\infty}^{\infty} d\omega e^{j\omega t} \mathbf{M}(\mathbf{r}, \omega) \tag{2.4}$$

Using the fact that

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} d\omega e^{j\omega t} \mathbf{B}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} d\omega \frac{\partial}{\partial t} e^{j\omega t} \mathbf{B}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} d\omega e^{j\omega t} j\omega \mathbf{B}(\mathbf{r}, \omega) \tag{2.5}$$

and that

$$\nabla \times \int_{-\infty}^{\infty} d\omega e^{j\omega t}\mathbf{E}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} d\omega e^{j\omega t}\nabla \times \mathbf{E}(\mathbf{r},\omega) \tag{2.6}$$

Furthermore, using the fact that

$$\int_{-\infty}^{\infty} d\omega e^{j\omega t}A(\omega) = \int_{-\infty}^{\infty} d\omega e^{j\omega t}B(\omega), \quad \forall t \tag{2.7}$$

implies that $A(\omega) = B(\omega)$, and using (2.5) and (2.6) in (2.4), and the property (2.7), one gets

$$\nabla \times \mathbf{E}(\mathbf{r},\omega) = -j\omega\mathbf{B}(\mathbf{r},\omega) - \mathbf{M}(\mathbf{r},\omega) \tag{2.8}$$

These equations look exactly like the phasor equations we have derived previously, save that the field $\mathbf{E}(\mathbf{r},\omega)$, $\mathbf{B}(\mathbf{r},\omega)$, and $\mathbf{M}(\mathbf{r},\omega)$ are now the Fourier transforms of the field $\mathbf{E}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$, and $\mathbf{M}(\mathbf{r},t)$. Moreover, the Fourier transform variables can be complex just like phasors. Repeating the exercise above for the other Maxwell's equations, we obtain equations that look similar to those for their phasor representations. Hence, Maxwell's equations can be simplified either by using phasor technique or Fourier transform technique.

3 Complex Power

Consider now that in the phasor representations, $\underset{\sim}{\mathbf{E}}(\mathbf{r})$ and $\underset{\sim}{\mathbf{H}}(\mathbf{r})$ are complex vectors, and their cross product, $\underset{\sim}{\mathbf{E}}(\mathbf{r}) \times \underset{\sim}{\mathbf{H}}^*(\mathbf{r})$, which still has the unit of power density, has a different physical meaning. First, consider the instantaneous Poynting's vector

$$\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t) \tag{3.1}$$

where all the quantities are real valued. Now, we can use phasor technique to analyze the above. Assuming time-harmonic fields, the above can be rewritten as

$$\begin{aligned} \mathbf{S}(\mathbf{r},t) &= \Re e[\underset{\sim}{\mathbf{E}}(\mathbf{r})e^{j\omega t}] \times \Re e[\underset{\sim}{\mathbf{H}}(\mathbf{r},)e^{j\omega t}] \\ &= \frac{1}{2}[\underset{\sim}{\mathbf{E}}e^{j\omega t} + (\underset{\sim}{\mathbf{E}}e^{j\omega t})^*] \times \frac{1}{2}[\underset{\sim}{\mathbf{H}}e^{j\omega t} + (\underset{\sim}{\mathbf{H}}e^{j\omega t})^*] \end{aligned} \tag{3.2}$$

where we have made use of the formula that

$$\Re e(Z) = \frac{1}{2}(Z + Z^*) \tag{3.3}$$

Then more elaborately, on expanding (3.2), we get

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{4}\underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}}e^{2j\omega t} + \frac{1}{4}\underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}}^* + \frac{1}{4}\underset{\sim}{\mathbf{E}}^* \times \underset{\sim}{\mathbf{H}} + \frac{1}{4}\underset{\sim}{\mathbf{E}}^* \times \underset{\sim}{\mathbf{H}}^* e^{-2j\omega t} \tag{3.4}$$

Then rearranging terms and using (3.3) yield

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{2}\Re e[\underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}}^*] + \frac{1}{2}\Re e[\underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}} e^{2j\omega t}] \tag{3.5}$$

where the first term is independent of time, while the second term is sinusoidal in time. If we define a time-average quantity such that

$$\mathbf{S}_{av} = \langle \mathbf{S}(\mathbf{r},t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{S}(\mathbf{r},t)dt \tag{3.6}$$

then it is quite clear that the second term of (3.5) time averages to zero, and

$$\mathbf{S}_{av} = \langle \mathbf{S}(\mathbf{r},t)\rangle = \frac{1}{2}\Re e[\underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}}^*] \tag{3.7}$$

Hence, in the phasor representation, the quantity

$$\underset{\sim}{\mathbf{S}} = \underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}} \tag{3.8}$$

is termed the complex Poynting's vector. The power flow associated with it is termed complex power.

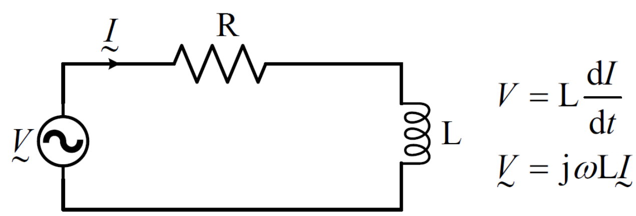

Figure 2:

To understand what complex power is , it is fruitful if we revisit complex power in our circuit theory course. The circuit in Figure 2 can be easily solved by using phasor technique. The impedance of the circuit is $Z = R+j\omega L$. Hence,

$$\underset{\sim}{V} = (R + j\omega L)\underset{\sim}{I} \tag{3.9}$$

Just as in the electromagnetic case, the complex power is taken to be

$$\underset{\sim}{P} = \underset{\sim}{V}\,\underset{\sim}{I}^* \tag{3.10}$$

$$P_{inst}(t) = V(t)I(t) \tag{3.11}$$

As shall be shown below, it is quite easy to that

$$P_{av} = \langle P_{inst}(t)\rangle = \frac{1}{2}\Re e[\underset{\sim}{P}] \tag{3.12}$$

where P_{inst} is the instantaneous power. It is clear that if $V(t)$ is sinusoidal, it can be written as

$$V(t) = V_0 \cos(\omega t) = \Re e[\underset{\sim}{V} e^{j\omega t}] \tag{3.13}$$

where we assume that $\underset{\sim}{V} = V_0$. Then from (3.9), it is clear that $V(t)$ and $I(t)$ are not in phase. Namely that

$$I(t) = I_0 \cos(\omega t + \alpha) = \Re e[\underset{\sim}{I} e^{j\omega t}] \tag{3.14}$$

where $\underset{\sim}{I} = I_0 e^{j\alpha}$. Then

$$\begin{aligned} P_{inst}(t) &= V_0 I_0 \cos(\omega t)\cos(\omega t + \alpha) \\ &= V_0 I_0 \cos(\omega t)[\cos(\omega t)\cos(\alpha) - \sin(\omega t)\sin\alpha] \\ &= V_0 I_0 \cos^2(\omega t)\cos\alpha - V_0 I_0 \cos(\omega t)\sin(\omega t)\sin\alpha \end{aligned} \tag{3.15}$$

It can be seen that the first term does not time-average to zero, but the second term does. Now taking the time average of (3.15), we get

$$P_{av} = \langle P_{inst} \rangle = \frac{1}{2} V_0 I_0 \cos\alpha = \frac{1}{2} \Re e[\underset{\sim}{V}\underset{\sim}{I}] \tag{3.16}$$

$$= \frac{1}{2} \Re e[\underset{\sim}{P}] \tag{3.17}$$

On the other hand,

$$P_{reactive} = \frac{1}{2} \Im m[\underset{\sim}{P}] = \frac{1}{2} \Im m[V_0 I_0 e^{j\alpha}] = \frac{1}{2} V_0 I_0 \sin\alpha \tag{3.18}$$

One sees that amplitude of the time-varying term in (3.15) is precisely proportional to $\Im m[\underset{\sim}{P}]$.

The reason for the existence of imaginary part of $\underset{\sim}{P}$ is because $V(t)$ and $I(t)$ are out of phase or $\underset{\sim}{V} = V_0$, but $\underset{\sim}{I} = I_0 e^{j\alpha}$. The reason why they are out of phase is because the circuit has a reactive part to it. Hence the imaginary part of complex power is also called the reactive power. In a reactive circuit, the plot of the instantaneous power is shown in Figure 3.

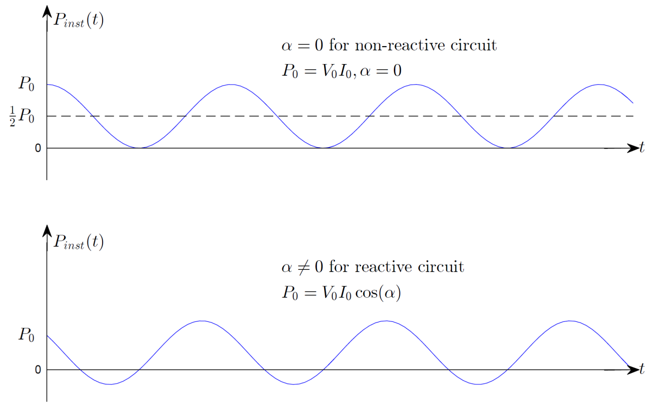
$P_{inst}(t)$
$\alpha = 0$ for non-reactive circuit
$P_0 = V_0 I_0, \alpha = 0$
P_0
$\frac{1}{2}P_0$
0
t
$P_{inst}(t)$
$\alpha \neq 0$ for reactive circuit
$P_0 = V_0 I_0 \cos(\alpha)$
P_0
0
t

Figure 3: